Int J Theor Phys (2010) 49: 1886–1893
DOI 10.1007/s10773-010-0371-9

String Theory in Embeddings Manifolds

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Received: 31 August 2009 / Accepted: 30 October 2009 / Published online: 18 May 2010

Abstract We consider the problem of infrared divergencies in String theory in Embeddings Manifolds by means of the Nash Theorem of Riemann metrics parametrized by immersions.

1 Introduction

In modern quantum field theory, the framework of strings moving in manifolds has been successfully used to shed light in the basic problem of quantizing the Gravitation field [1]. Moreover, until now the severe problems of the infrared divergencies of the string theory path integral when viewed as a σ-model two-dimensional field theory in the parameter string domain R^2 has been an issue not completely understood [2–4]. Although there is a strong indication that it is possible to remove such quantum field theoretic difficulties of the use of a mathematically ill-defined 2D-massless quantum scalar-field [represented by the string vector position] by means of a string third quantization (the so called String Field Theory), this step remains an unsolved problem in the present framework of String Theory.

The purpose of this paper is to consider another framework for the problem of the infrared divergences in String Theory by applying the Nash theorem of Riemann metrics parametrized by immersions in order to show the appearance of a string mass effective matrix as a result of the dynamical interaction with the positive curvature of the given string ambient space-time M, when considered as a smooth C^∞-differentiable manifold.

2 The String Mass from the Smooth Space-Time Manifold Shape-Bending in the Extrinsic Space

Let us start our analysis by considering the following convenient Euclidean Polyakov's string functional integral in the presence of a given back-ground fixed Riemannian metric in

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the manifold M where the string dynamics takes place.

$$Z = \left\{ \int d^{\text{cov}}\mu[g_{ab}(\xi)] d^{\text{cov}}\mu[X^\mu(\xi)] \times \exp\left\{ -\frac{1}{2\pi\alpha'} \int_{R^2} d^2\xi \{\sqrt{g} g^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu}(X(\xi))\} \times \left\{ \prod_{\ell=1}^{(s(d)-d)} \delta^{(F)}\big(H^\ell(f^A(X^B(\xi)))\big) \right\}\right\} \tag{1}$$

the (closed) string surface $\{X^\mu(\xi), \mu = 1, \dots, d\}$ is immersed in the space-time M given by a manifold possessing a $C^\infty(M)$-smooth Riemannian structure (metric) $\{G_{\mu\nu}(x^\gamma)\}_{\substack{\mu=1,\dots,d \\ \nu=1,\dots,d}}$ The manifold parametric explicit set of equations is denoted here by $H^\ell(f^A(x^B)) \equiv 0$, $A = 1, \dots, s(d))$ and $f^A\colon M \to R^{s(d)}$ is the set of real-valued immersions such that we have for them the Nash theorem for our smooth given space-time manifold metric $\{G_{\mu\nu}(x^\gamma)\}$. Namely [5]

$$G_{\mu\nu}(x^\gamma) = \sum_{A=1}^{s(d)} \left[\frac{\partial f_A}{\partial x^\mu} \frac{\partial f_A}{\partial x^\nu} \right](x^\gamma). \tag{2}$$

Here $s(d)$ is the minimal Whitney immersion dimension of the manifold M in $R^d(S(d) > 2d)$.

The covariant functional measures in the Polyakov path integral (1) are the well-known De-Witt covariant functional metrics without boundary terms. Namely:

$$dS^2[g_{ab}] = \int_{R^2} d^2\xi \left[\sqrt{g}(\delta g_{ab})[g^{aa'}g^{bb'} + c g^{ab} g^{a'b'}](\delta g_{a'b'}) \right](\xi), \tag{3}$$

$$dS^2[X^\mu] = \int_{R^2} d^2\xi \left[\sqrt{g}\delta X^\mu(\xi) G_{\mu\nu}(X^\gamma(\xi)) \delta X^\nu(\xi) \right](\xi). \tag{4}$$

Let us show the announced phenomenon of geometrical mass generation for the 2D-scalar string vector-position fields $\{X^\mu(\xi), \xi = 1, \dots, d\}$, in the situation of a weakly space-time manifold of positive curvature.

Our main propose is to consider the following variable change in the string vector position dynamical degree of freedom (see (2)) in the full String Partition Functional Path Integral (1).

$$Y^A(\xi) = f^A(X^\mu(\xi)), \quad A = 1, \dots, s(d), \tag{5a}$$

$$S[Y^A(\xi)] = \frac{1}{2\pi\alpha'} \int_{R^2} d^2\xi \sqrt{g} g^{ab} (\partial_a Y^A \partial_b Y_A)(\xi), \tag{5b}$$

$$dS^2[(Y^A(\xi)] = \int_{R^2} d^2\xi [\sqrt{g}\delta Y^A \delta Y_A](\xi). \tag{5c}$$

At this point of our study we point at the usefulness on the explicitly use of the geometrical constraint that the string world-sheet Σ is in M through the writing of the supposed known set of the Space-Time Manifold parametric equations $\{H^\ell(Y^A) = 0$, $\ell = 1, \dots, s(d) - d, \{Y^A\} \in M\}$ defining M as an embedding geometrical-positional submanifold of the (Absolute-Extrinsic) Euclidean Whitney Space $R^{s(d)}$. This last step is the basic mechanism for our proposal of generating mass for the mean effective string vector position $\{Y^A(\xi), A = 1, \dots, s(d), \xi \in R^2\}$.

In order to show these string mass generation mechanism by geometric means, let us suppose that we have a manifold with very low positive curvature.

In this case we can replace the delta functional geometrical constraint in (5) by the effective string mass term as written below

$$\prod_{\ell=1}^{(s(d)-d)} \delta^{(F)}_{\text{cov}}\big(H^\ell(Y^A(\xi))\big) \underset{\substack{\text{(low extrinsic}\\ \text{curvature)}}}{\cong} \prod_{\ell=1}^{(s(d)-d)} \delta^{(F)}_{\text{cov}}\left[\left(\frac{\alpha'}{2}\frac{\partial^2 H^\ell}{\partial Y^A \partial Y^B}(\overline{Y}^c)\widetilde{Y}^A\widetilde{Y}^B\right)(\xi)\right], \tag{6}$$

where we have used the zero mode of mean string vector position variable in terms of the constant mode $\overline{Y}$ and its α'-vanishing small fluctuation

$$Y^A(\xi) = \overline{Y}^A + \sqrt{\pi\alpha'}\widetilde{Y}^A(\xi). \tag{7}$$

By making the usual hypothesis of the exact validity of the covariant mean field average for the Lagrange multiplier in the Path-Integral representation for the effective functional delta (6), we get the following explicitly results

$$\begin{aligned} &\prod_{\ell=1}^{s(d)-d} \delta^{(F)}\left[\left(\alpha'\frac{\partial^2 H^\ell}{\partial Y^A \partial Y^B}(\overline{Y}^c)\widetilde{Y}^A\widetilde{Y}^B\right)(\xi)\right] \\ &\cong \prod_{\ell=1}^{s(d)-d}\left[\int d^{\text{cov}}\mu[\lambda^\ell(\xi)]\exp\left\{i\alpha'\int_D d^2\xi\sqrt{g}\left[\lambda_\ell\left(\frac{1\partial^2 H^\ell(\overline{Y}^c)}{2\partial Y^A\partial Y^B}\widetilde{Y}^A\widetilde{Y}^B\right)\right](\xi)\right]\right\} \\ &\sim \exp\left\{-\mu_{AB}(\overline{Y}^c)\int_D d^2\xi(\sqrt{g}\widetilde{Y}^A\widetilde{Y}^B)(\xi)\right\}. \end{aligned} \tag{8}$$

Here the string mass matrix is given explicitly by the combination of the curvature position Hessian Space-Time manifold matrix at the point $\{\overline{Y}^c\} \in M$ and the (positive) condensate value of Lagrange multiplier field $\lambda^\ell_{ab}(\xi) \cong i\langle\lambda\rangle\delta_{ab}$, producing thus the result

$$\mu_{AB}(\overline{Y}^c) = \frac{1}{2}\langle\lambda\rangle\left(\frac{\partial^2 H^\ell}{\partial Y^A\partial Y^B}(\overline{Y}^c)\right). \tag{9}$$

At this point appears worthing mentioning that the non-linearity of the original theory fully appears as a consequence of the highly non-trivial re-writing of the string vertexs in terms of the somewhat decoupling-ambient geometry (5a).[1]

Now we proceed to the Nambu-Goto string path integral which depends functionally solely on the string world sheet imbedding $X^\mu(\xi)\colon R^2 \to R^D$, namely

$$Z = \int d_h\mu[X^\mu(\xi)]\exp\left\{-\frac{1}{2\pi\alpha'}\int_{R^2} d^2\xi\left(\sqrt{h(X^\alpha(\pi))}\right)\right\}. \tag{10}$$

[1] The scalar vertex, for instance is given by

$$V^{\text{sc}}(x^A) := \delta^{(d)}(x^A - X^A(\xi)) = \delta^{(d)}(x^A - [(f^A)^{-1}(Y^B)](\xi)).$$

Here the string world sheet metric tensor is always given by the imbedding variable $X^\mu(\xi)$

$$h_{ab}(X^\alpha(\xi)) = \partial_a X^\mu(\xi) G_{\mu\nu}(X^\beta(\xi))\partial_b X^\nu(\xi). \tag{11}$$

In this string theory, the main difficulty comes from the diffeomorphism invariant measure $D^{\text{cov}}[X^\mu(\xi)]$ which is strongly non-linear when written as a Feynman product measure as given below

$$d_h\mu[X^\mu(\xi)] = \prod_{\xi\in R^2}\left[\big(h(X^\mu(\xi))\big)^{1/4}\big(G(X^\mu(\xi))\big)^{1/2} dX^\mu(\xi)\right], \tag{12a}$$

$$d^2S[X^\mu(\xi)] = \int_{R^2} d^2\xi \sqrt{h(X^\alpha(\xi))}\big(\delta X^\mu G_{\mu\nu}(X^\gamma)\delta X^\nu\big)(\xi). \tag{12b}$$

In order to overcome such problem, we proceed as in [6] by considering the 2D-fluctuating metric tensor fields $g_{ab}(\xi)$ as a purely auxiliary Lagrange multiplier field without any singled out geometrical-physical role and whose dynamics must be suppressed at the end of the path integrals evaluations[2]

$$\begin{aligned} Z = \int d\mu[g_{ab}(\xi)] \int d\mu[X(\xi)] \exp\left\{-\frac{1}{2\pi\alpha'}\int_{R^2} d^2\xi(\sqrt{g})(\xi)\right\} \\ \times \exp\left\{-\frac{1}{2}\int_{R^2} d^2\xi(\sqrt{g}g^{ab}\partial_a X^\mu G_{\mu\nu}(X)\partial_b X^\nu)(\xi)\right\} \\ \times \delta^{(F)}_{\text{cov}}\big([g_{ab} - (\partial_a X^\mu \partial_b X^\nu G_{\mu\nu}(X)](\xi)\big). \end{aligned} \tag{13}$$

It is worth call the reader attention that the original Polyakov's propose (1) must be considered as an effective (analytical) path integral procedure in the light of the Nash Theorem when applied to the string world sheet as a two-dimensional manifold immersed (not fully embedded) in R^D $(0 \geq 4)$ since there is a clear over counting of the degrees of freedom in (1) parametrizing the string dynamics: For each two-dimensional metric field $g_{ab}(\xi)$ in the string world sheet tangent bundle there is an immersion $X^\mu(\xi,[g])\colon \Sigma \to R^J$, in some Whitney ambient space $R^{\bar{d}}$ $(\bar{d} > 3)$ and satisfying the metrical constraint

$$g_{ab}(\xi) = \frac{\partial X^\mu(\xi,[g])}{\partial \xi^a}\frac{\partial X_\mu(\xi,[g])}{\partial \xi^b}. \tag{14}$$

As a consequence of the above remark, one can see that our propose (13) already takes into account this deep geometrical-topological constraint between the string world-sheet metrical fields and the immersion/string vector position in the extrinsic space in a correct mathematical way by means of the (covariant) delta functional inside (13).

[2]Note the path integral identity

$$\begin{aligned} \int d^{\text{cov}}_\mu[g_{ab}]\int d^{\text{cov}}_\mu[X]\exp\left\{-\frac{1}{2\pi\alpha'}\int_{R^2}(\sqrt{g}g^{ab}\partial_a X^\mu \partial_b X_\mu(\xi)d^2\xi\right\} \\ \times \delta^{(F)}_{\text{cov}}(g_{ab} - \partial_a X^\mu \partial_b X_\mu) = \int d^{\text{cov}}\mu[X^\mu]\exp\left(-\frac{1}{2\pi\alpha'}\int_{R^2}\sqrt{h}(\xi)d^2\xi\right). \end{aligned}$$

By proceeding again as in [6] we can evaluate the covariant path integrals in (13) in terms of the usual Feynman product measures in the light-cone gauge

$$Z = \int D^F[Y^A(\xi)] \exp\left\{-\frac{1}{2\pi\alpha'}\int_{R^2} d\xi^+ d\xi^- \big[(\partial_+ Y^A \partial_- Y_A)(\xi^+,\xi^-)\big]\right\}$$
$$\times \exp\left\{-\frac{(26-s(D))}{48\pi}\int_{R^2} d\xi^+ d\xi^- \left[\frac{[\partial_+(\partial_+ Y^A\partial_- Y_A)\times\partial_-(\partial_+ Y^B\partial_- Y_B)]}{(\partial_+ +_Y^B \partial_- Y_B Y^A \partial_- Y_A)^2}\right](\xi^+,\xi^-)\right\}$$
$$\times \exp\left\{-\frac{1}{2}\int_{R^2} d\xi^+ d\xi^- (\mu_{AB}(\gamma)(Y^A Y_B)(\xi))\right\}. \quad (15)$$

The introduction of non-trivial topology in the string world sheet is now straightforward in our Path-Integral analysis and the suppression of the Liouville dynamics for the unphysical field $g_{ab}(\xi)$ in the Polyakov scheme can be made by introducing N fermion species in order to change the conformal anomaly coefficient to the new factor $\frac{26-(S(D)+N)}{48}$, which can vanishes if one choose $26 = S(D) + N$.

As a last remark in this section, let us point out that in the case of a compact string parameter domain $D \subset R^2$ (not the fully R^2), one should in principle introduces in the path integral (8)/(10) a further sum over these domains, in order to obtain full covariance. For instance, if one choose the rectangle $D_A = \{(\xi_1,\xi_2), 0 \le \xi_1 \le A; 0 \le \xi_2 \le 2\pi\}$, one should introduce a further integration in relation to the “moduli” A, namely

$$Z = \int_0^\infty dA \left\{\int d_h\mu[X^\mu(\xi)] \exp\left[-\frac{1}{2\pi\alpha'}\int_{D_A} d^2\xi(\sqrt{h}(X^\mu(\xi)))\right]\right\}. \quad (16)$$

Note that the Green function associated to the compact domain D_A does not posseses infrared divergencies as in R^2, as one can see for its explicitly expression below

$$\langle X^\mu(z,\bar z\rangle X^\nu(\zeta,\bar\zeta)\rangle_{D_A}$$
$$= \left(-\frac{1}{2\pi}\mathrm{Re}\left\{\log\left[\frac{\sigma(z-\zeta,w_1,w_2)\sigma(z+\zeta,w_1,w_2)}{\sigma(z-\bar\zeta,w_1,w_2)\sigma(z+\bar\zeta,w_1,w_2)}\right]\right\}\right)\delta^{\mu\nu}. \quad (17)$$

Here

$$z = x + iy, \qquad \zeta = \xi + i\eta$$
$$w_1 = A, \qquad w_2 = 2\pi$$

and the Weierstrass-Elliptic σ-function has the expression

$$\sigma(z) = z\prod_w \left[\left(1-\frac{z}{2w}\right)e^{(\frac{3}{2w}+\frac{z^2}{8w^2})}\right],$$
$$w = kA + 2\pi\ell i \ (k = 0, \pm 1, \dots); \ (\ell = 0, \pm 1, \dots).$$

The reader should compare with the infrared divergent String Green function in the full space R^2

$$\langle X^\mu(z,\bar z)X^\nu(\zeta,\bar\zeta)\rangle = \delta^{\mu\nu}\left(-\frac{1}{4\pi}\ell g|z-\zeta|\right). \quad (18)$$

3 The Einstein-Hilbert Action as an Effective Theory for Random (Stringy) Fluctuations of the Space-Time

In this somewhat section, we intend to show how the Einstein-Hilbert action for Einstein Gravitation Theory appears in a rather natural way from a Bosonic Polyakov's String interacting with the ambient (extrinsic) manifold fluctuating metrical structure.

Let us thus firstly write the Polyakov's string path integral in the presence of the metric tensor $G_{\mu\nu}(X^\alpha)$:

$$Z[G_{\mu\nu}(X^\alpha(\xi)] = \int \left[\prod_{\xi\in R^2} \left(\sqrt{G}(X^\mu(\xi))\right)^{1/2} dY^\mu(\xi)\right] \times \exp\left\{-\frac{1}{2\pi\alpha'}\int_{R^2} d^2\xi\left[(G_{\mu\nu})(Y^\beta)\partial_a Y^\mu \partial^a Y^\nu\right](\xi)\right\}. \tag{19}$$

In order to see how (Higher order) Einstein-Hilbert actions emerges as an effective theory from (1), let us consider the geodesic expansion for the metrical objects in (1) through a power series expansion in the string lenght extrinsic scale α'. (Here $\sigma^{\alpha\beta}(\xi) = (X^\alpha X^\beta)(\xi)$):

$$Y^\mu(\xi) = \overline{Y}^\mu + \sqrt{\alpha'} X^\mu(\xi), \tag{20}$$

$$\begin{aligned}\sqrt{G(Y^\mu(\xi))} &= 1 - \frac{\alpha'}{6} R_{\mu\nu}(\overline{Y}^B)\cdot(\sigma^{\mu\nu})(\xi) - \frac{(\alpha')^{3/2}}{12}(\nabla_\alpha R_{\mu\nu})(\overline{Y}^\beta)(X^\alpha\sigma^{\mu\nu})(\xi) \\ &\quad + \frac{(\alpha')^2}{24}\left\{\left[-\frac{3}{5}(\nabla_\mu\nabla_\nu R_{\alpha\beta})(\overline{Y}^\beta) + \frac{1}{3}(R_{\mu\nu}R_{\alpha\beta})(\overline{Y}^\beta)\right.\right. \\ &\quad \left.\left. - \frac{2}{15} R_{\mu\sigma\nu w}(\overline{Y}^\beta) R_{\alpha\sigma\beta w}(\overline{Y}^\beta)\right](\sigma^{\mu\nu}\sigma^{\alpha\beta})(\xi)\right\} + O((\alpha')^{2+n}),\end{aligned} \tag{21}$$

$$\begin{aligned}G_{\mu\nu}(Y^\mu(\xi)) &= \delta_{\mu\nu} - \frac{(\alpha')}{3} R_{\alpha\mu\beta\nu}(\overline{Y})(X^\alpha X^\beta)(\xi) \\ &\quad - \frac{1}{6}(\alpha')^{3/2}(\nabla_\alpha R_{\beta\mu\sigma\nu}(\overline{Y}^\beta)(X^\alpha X^\beta X^\sigma)(\xi) \\ &\quad + \frac{(\alpha')^2}{36}\left\{\left[-18\nabla_\alpha\nabla_\beta R_{w\mu\sigma\nu} + 16 R_{\alpha\mu\beta\gamma} R_{w\nu\sigma\gamma}\right](\overline{Y}^\beta)\right. \\ &\quad \left.\times (X^\alpha X^\beta X^w X^\sigma)(\xi)\right\} + O((\alpha')^{2+n}).\end{aligned} \tag{22}$$

At this point let us re-write (1) in terms of the composite operator $\sigma^{\alpha\beta}(\xi) = (X^\alpha X^\beta)(\xi)$ by considering the identity insertion

$$\begin{aligned}&\delta^{(F)}\left(\sigma^{\alpha\beta}(\xi) - (X^\alpha X^\beta)(\xi)\right) \\ &= \int \left(\prod_{\xi\in R^2} d\lambda(\xi)\right)\exp\left\{i\int_{R^2} d^2\xi\, G_{x\beta}(\overline{Y})X\lambda(\xi)\right. \\ &\quad \left.\times[\sigma^{x\beta}(\xi) - (X^x X^\beta)(\xi)]\right\} \\ &\cong \exp\left\{-\langle\lambda\rangle\int_{R^2} d^2\xi\, G_{x\beta}(\overline{Y})[\sigma^{x\beta}(\xi) - (X^x X^\beta)(\xi)\right\}.\end{aligned} \tag{23}$$

As a consequence we have the result

$$Z[G_{\mu\nu}(X^\alpha)] = \prod_{\overline{Y}\in M} \widetilde{Z}[G_{\mu\nu}(\overline{Y})] \tag{24}$$

with

$$\begin{aligned}
&\widetilde{Z}[G_{\mu\nu}(\overline{Y})] \\
&\quad = \int \prod_{\xi\in R^2} (\sqrt{G}(\overline{Y}))^{1/2} dX(\xi) \\
&\qquad \times \exp\left\{-\frac{1}{2\pi\alpha'}\int_{R^2} d^2\xi \big(\text{eq (21)}\big)\big(\text{eq (22)}\big)\partial_a X^\mu \partial^a X^\nu\right\} \qquad (25)\\
&\quad = \det{}^{-\frac{1}{2}}\Bigg[\bigg(\delta_{\mu\nu} + \frac{\alpha'\langle\sigma\rangle}{3} R_{\alpha\mu\alpha\nu}(\overline{Y}) \\
&\qquad + \frac{4}{9}(\alpha')^2\langle\sigma\rangle^2 \big(R_{\alpha\mu\alpha\gamma} R^\gamma_{\beta\nu\beta})(\overline{Y}) + \cdots)(-\partial_a\partial^a)_\xi + \langle\lambda\rangle\delta_{\mu\nu}\Bigg] \\
&\qquad \times \exp\left\{-\frac{1}{2}\int_{R^2\times R^2} d^2\xi d^2\xi' \left[-\frac{(\alpha')^{3/2}}{12}(\nabla_\alpha R_{\mu\nu}(\overline{Y}) + \cdots\right]_{\mu\zeta}\right. \\
&\qquad \times \left[\left(\delta_{\mu\nu} + \frac{\alpha'\langle\sigma\rangle}{3} R_{\alpha\mu\alpha\nu} + \cdots\right)(-\partial_a\partial^a)_\xi + \langle\lambda\rangle\delta_{\mu\nu})\right]^{-1}_{\zeta\sigma'}(\xi,\xi') \\
&\qquad \times \left.\left[-\frac{(\alpha')^{3/2}}{12}(\nabla_\alpha R_{\mu\mu})(\overline{Y}) + \cdots\right]_{\sigma'\mu}\right\}, \qquad (26)
\end{aligned}$$

where we have supposed another time the condensate formation for the bilinear field $\sigma^{\alpha\beta}(\xi) = \langle\sigma\rangle G^{\alpha\beta}(\overline{Y})$ and the implicity use of the saddle-point limit of $\langle\lambda\rangle \to \infty$ for the Lagrange multiplier.

At this point and for pedagogical purpose let us evaluate the following sample calculations of (26).

$$\begin{aligned}
&\lim_{\left\{\substack{\alpha'\to 0\\ \langle\lambda\rangle\to\infty}\right\}} \left\{\det{}^{-\frac{1}{2}}\left[\left(\delta_{\mu\nu} + \frac{\langle\sigma\rangle}{3}(\alpha')R_{\mu\alpha\nu\alpha}(\overline{Y})\right)(-\partial^2)_\xi + \langle\lambda\rangle\delta_{\mu\nu}\right]\right\} \\
&\quad = \lim_{\left\{\substack{\alpha'\to 0\\ \langle\lambda\rangle\to\infty}\right\}} \det{}^{-\frac{1}{2}}\left[(-\partial^2)_\xi\delta_{\mu\nu} + \langle\lambda\rangle\left(\delta_{\mu\nu} - \frac{\langle\sigma\rangle}{3}\alpha' R_{\mu\alpha\nu\alpha}(\overline{Y})\right)\right]. \qquad (27)
\end{aligned}$$

Now one can see (details as exercise for our readers)

$$\begin{aligned}
&\log\det{}^{-\frac{1}{2}}\left[(-\partial^2)_\xi\delta_{\mu\nu} + \langle\lambda\rangle\left(\delta_{\mu\nu} - \frac{\langle\sigma\rangle}{3}\alpha' R_{\mu\alpha\nu\alpha}(\overline{Y})\right)\right] \\
&\quad = \lim_{\left\{\substack{\alpha'\to 0\\ \langle\lambda\rangle\to\infty}\right\}} \int_\varepsilon^\infty \frac{dt}{t} e^{-t\langle\lambda\rangle} \cdot \operatorname{Tr}\exp\left\{-t\left[(-\partial^2)_\xi\delta_{\mu\nu} + \langle\lambda\rangle\left(\delta_{\mu\nu} - \frac{\langle\sigma\rangle}{3}\alpha' R_{\mu\alpha\nu\alpha}(\overline{Y})\right)\right]\right\} \\
&\quad = \int_\varepsilon^\infty \frac{dt}{t} e^{-t\langle\lambda\rangle} \lim_{t\to 0^+} \operatorname{Tr}\exp\{-t \text{ [above written operator]}
\end{aligned}$$

$$\sim \sqrt{G(\overline{Y})}\left\{c_0(\varepsilon)\int_\varepsilon^\infty \frac{dt}{t}e^{-t\langle\lambda\rangle}\right\}$$
$$-\sqrt{G(\overline{Y})}R(\overline{Y})\left\{\frac{\langle\lambda\rangle\langle\sigma\rangle\alpha'}{3}c_1(\varepsilon)\int_\varepsilon^\infty \frac{dt}{t}e^{-1\langle\lambda\rangle}\right\}+O((\alpha')^2). \quad (28)$$

After inserting (9) into (6), we get as the leading limit of $\alpha' \to 0$ of the String Theory (1), the Einstein-Hilbert action with an effective cosmological constant and Newton Gravitation constant

$$\tilde{Z}[G_{\mu\nu}(\overline{Y})] = \exp\left\{-\mu^{\mathrm{eFF}}\int_M d\overline{Y}\sqrt{G(\overline{Y})} - \frac{1}{8\pi G_N^{\mathrm{eFF}}}\int_N d\overline{Y}\sqrt{G(\overline{Y})}R(\overline{Y})\right\}, \quad (29)$$

where

$$\mu^{\mathrm{eFF}}(\xi) \sim A\int_\varepsilon^\infty \frac{dt}{t^2}e^{-t\langle\lambda\rangle}, \quad (30)$$
$$\frac{1}{8\pi G_N^{\mathrm{eFF}}} \underset{(\varepsilon)}{\sim} A\left(\int_\varepsilon^\infty \frac{dt}{t}e^{-t\langle\lambda\rangle}\right)\left(\frac{\langle\lambda\rangle\langle\sigma\rangle\alpha'}{3}\right), \quad (31)$$
$$A = \int_{R^2} d^2\xi.$$

If one consider the fluctuations of our metrical tensor $G_{\mu\nu}(\overline{Y})$ on M, one should consider a further path-integral on (1) relation to the dynamical external field $G_{\mu\nu}(x^\gamma)$, and thus getting quantum gravity as an effective quantum path integral theory.[3]

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[3] For the Nambu-Goto fermiomic action, one should consider the fermionized area action

(a) $S := \frac{1}{2\pi\alpha'}\int_{R^2}(\sqrt{H})(\xi)d^2\xi$

(b) $H(\xi) = \det\{g_{ab}^F(\xi)\}$

(c) $g_{ab}^F(\xi) := \partial_a X^\mu \partial_b X_\mu + \frac{\ell}{2i}[\Psi^\mu(\gamma_\alpha\nabla_\beta - \nabla_\beta\gamma_\alpha)\Psi_\mu](\xi)$.